Complexity of Teaching by a Restricted Number of Examples

Hayato Kobayashi and Ayumi Shinohara

Tohoku University, Japan

Background

- Computational **teaching** theory
  - Aims to bring out the nature of teaching
  - which is inextricably linked to learning
    - Teachability [Shinohara and Miyano 1991]
    - Teaching dimension [Goldman and Kearns 1991]

    …
    - Expected teaching dimension [Balbach 2005]
    - Recursive teaching dimension [Zilles et al. 2008]

Censored

Illustrative problem: Phone-a-friend lifeline

"Who Wants to Be a Millionaire?"

- Millionaire (="Who Wants to Be a Millionaire?")
  - Challenge multiple-choice questions
  - Win a cash award depending on the number of correct answers
  - Get help from the three lifelines during the game

Censored

- Lifelines
  - **Phone-a-friend**
    - Will give you advice from friends
  - 50:50
    - Removes two incorrect answers
  - Ask the Audience
    - Lets you see the answers of audience

Censored

Classical model (1/3)

- Millionaire (one correct choice)

When was the COLT conference first held?

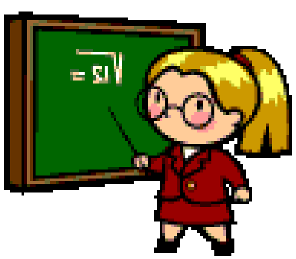
A: 1983 B: 1988

C: 1992 D: 1997

Concept class (available answers)
C = {{A}, {B}, {C}, {D}}

Learner (challenger)

Teaching



Teacher (friend)

Classical model (1/3)

- Millionaire (one correct choice)

When was the COLT conference first held?

A: 1983 | B: 1988
C: 1992 | D: 1997

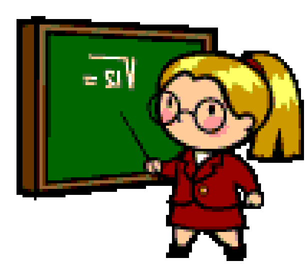
Concept class (available answers)
C = {{A}, {B}, {C}, {D}}

Target concept (correct answer)
c = {B}

Learner (challenger)

S = {(A, False), (B, True),
(C, False), (D, False)}

Teaching



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Classical model (1/3)

- Millionaire (one correct choice)

When was the COLT conference first held?

A: 1983

B: 1988

C: 1992

D: 1997

Concept class (available answers)
C = {{A}, {B}, {C}, {D}}

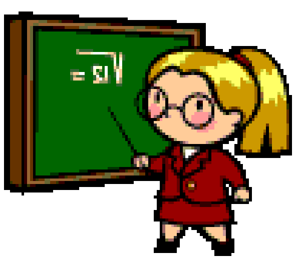
Target concept (correct answer)
c = {B}

Teaching set

S = {(A, False), (B, True), (C, False), (D, False)}

Learner (challenger)

Teaching



Teacher (friend)

Classical model (1/3)

- Millionaire (one correct choice)

When was the COLT conference first held?

A: 1983

B: 1988

C: 1992

Concept class (available answers)
$\boldsymbol{C}$ = {{A}, {B}, {C}, {D}}

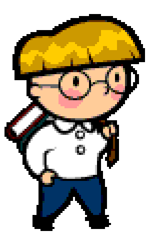
Teaching Dimension
TD(c, $\boldsymbol{C}$) := |Minimum teaching set|
In this case, TD(c, $\boldsymbol{C}$)=1

c = {B}

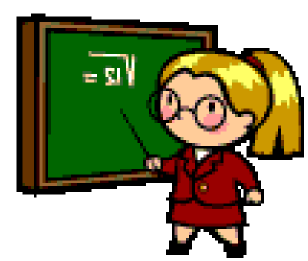
Teaching se

S = {(A, False), (B, True),
(C, False), (D, False)}

Teaching



Learner (challenger)



Teacher (friend)

Classical model (2/3)

- Millionaire 2.0 (two correct choices)

Which are the two cities where the Olympic games were held in Canada?

A: Montreal

B: Calgary

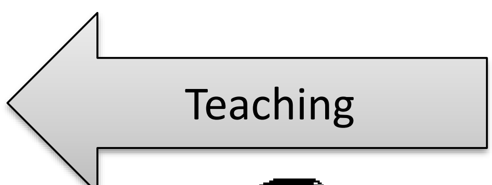
C: Ottawa

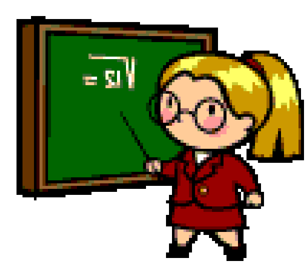
D: Vancouver

Concept class (available answers)

C = {{A,B}, {A,C}, {A,D},
{B,C}, {B,D}, {C,D}}

Learner (challenger)





Teacher (friend)

Classical model (2/3)

- Millionaire 2.0 (two correct choices)

Which are the two cities where the Olympic games were held in Canada?

A: Montreal

B: Calgary

C: Ottawa

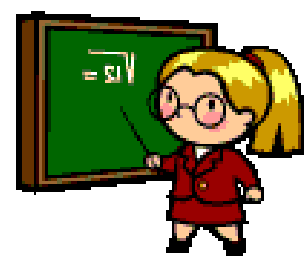
D: Vancouver

Concept class (available answers)

C = {{A,B}, {A,C}, {A,D}, {B,C}, {B,D}, {C,D}}

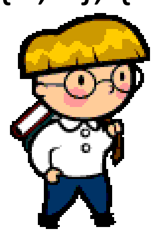
Target concept (correct answer)

c = {A, B}



S = {(A, True), (B, True)}

Teaching



Learner (challenger)

Teacher (friend)

Classical model (2/3)

- Millionaire 2.0 (two correct choices)

Which are the two cities where the Olympic games were held in Canada?

A: Montreal

B: Calgary

C: Ottawa

Teaching Dimension

TD(c, ***C***) := |Minimum teaching set|

In this case, TD(c, ***C***)=2

Concept class (available answers)

C = {{A,B}, {A,C}, {A,D},
{B,C}, {B,D}, {C,D}}

r)

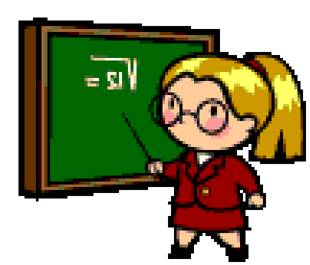
c = {A, B}

Minimum teaching set

S = {(A, True), (B, True)}

Learner (challenger)

Teaching



Teacher (friend)

Classical model (3/3)

- Generalized Millionaire (unknown # of correct choices)

Which choices are correct specialties of Canada?

A: Maple tea

B: Maple butter

C: Maple dressing

D: Maple mustard

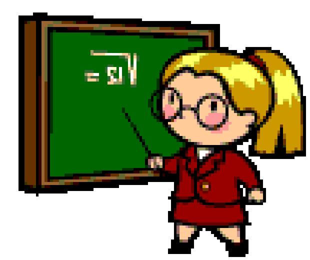
Concept class (available answers)
$\boldsymbol{C} = 2^{\{A,B,C,D\}}$

Target concept (correct answer)
c = {A, B, C, D}

Learner (challenger)

S = {(A, True), (B, True), (C, True), (D, True)}

Teaching



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Target concept (correct answer)
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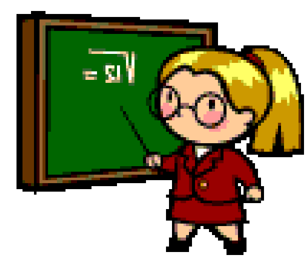
Learner (challenger)

S = {(A, True), (B, True),
(C, True), (D, True)}

Teaching



Teacher (friend)

Classical model (3/3)

- Generalized Millionaire (unknown # of correct choices)

Which choices are correct specialties of Canada?

A: Maple tea

C: Maple dressing

Teaching Dimension
TD(c, ***C***) := |Minimum teaching set|
In this case, TD(c, ***C***)=4

Concept class (available answers)
C = $2^{\{A,B,C,D\}}$

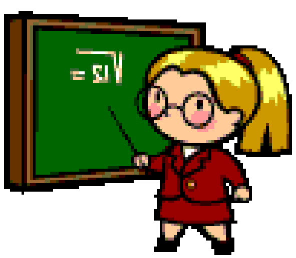
rget concept (correct answer)
c = {A, B, C, D}

Minimum teaching set

S = {(A, True), (B, True), (C, True), (D, True)}

Learner (challenger)

Teaching



Teacher (friend)

Our contributions

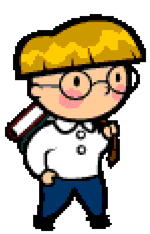
- **When # of examples < teaching dimension**
- Formal proofs that
  - Special teaching strategies are necessary
    - A subset of a teaching set is not always optimal
  - Smart teachers dare to tell a lie
    - Inconsistent examples are more useful
- Exact analyses of *<u>optimal teaching errors</u>* and *<u>optimally incremental teachabilities</u>* for concept classes of
  - M_n^+ : Monotone monomials
  - M_n' : Monomials without the empty concept
  - M_n : Monomials

Our model

- Restriction: # of examples ≦ k
  - Phone-a-friend lifeline: 30 sec.
  - This presentation: 25 min.
  - Lectures in our univ.: 90 min.

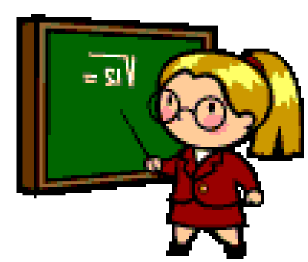
Millionaire 2.0 (two correct choices)
Concept class (available answers)
C = {{A,B}, {A,C}, {A,D},
{B,C}, {B,D}, {C,D}}

Target concept (correct answer)
c = {A, B}



Learner (challenger)

Teaching



Teacher (friend)

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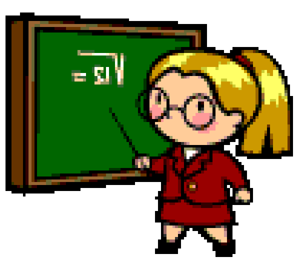
Millionaire 2.0 (two correct choices)
Concept class (available ar
C = {{A,B}, {A,C}, {A,D},
{B,C}, {B,D}, {C,D}}

"The question is
which two are …"
(He used 29 sec.)

Target concept (correct answer)
c = {A, B}

Learner (challenger)

Teaching



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  - This presentation: 25 min.
  - Lectures in our univ.: 90 min.

Millionaire 2.0 (two correct choices)

Concept class (available ar

C = {{A,B}, {A,C}, {A,D}, {B,C}, {B,D}, {C,D}}

"The question is which two are …"
(He used 29 sec.)

Target concept (correct answer)

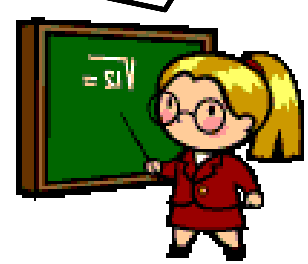
c = {A

"**A is True**. B is"

Learner (challenger)

S = {(A, True)}

Teaching



Teacher (friend)

Our model

- Restriction: # of examples ≦ k
  - Phone-a-friend lifeline: 30 sec.
  - This presentation: 25 min.
  - Lect... 0 min.

(Complexity)
Optimal Teaching Error

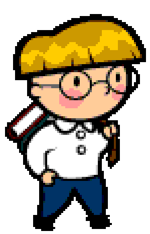
Millionaire 2.0 (two ... ect choices)
Concept class (available a...

$\boldsymbol{C}$ = {$\boxed{\{A,B\}, \{A,C\}, \{A,D\}}$,
~~{B,C}, {B,D}, {C,D}}~~

"The question is which two are …"
(He used 29 sec.)

Target concept (correct answer)
c = {A...

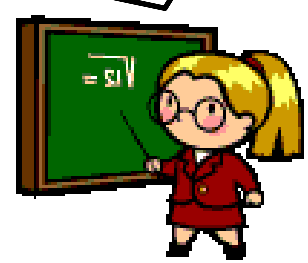
"**A is True**. B is"



Learner (challenger)

S = {(A, True)}

Teaching



Teacher (friend)

Optimal Teaching Error

Definition

Worst case error

$$OptTErr_k(c,C) := \min_{S:|S|\le k} \boxed{\max_{h\in CONS(S,C)} Err(c,h)}$$

k-optimal teaching sets achieving the optimal teaching error

$$OptTSets_k(c,C) := \arg\min_{S:|S|\le k} \max_{h\in CONS(S,C)} Err(c,h)$$

	h	A	B	C	D	Err(c, h)
c =	{A,B}	T	T	F	F	0/4
	{A,C}	T	F	T	F	2/4
	{A,D}	T	F	F	T	2/4
	{B,C}	F	T	T	F	2/4
	{B,D}	F	T	F	T	2/4
	{C,D}	F	F	T	T	4/4

$$Err(c,h) := \frac{|c \Delta h|}{|X|} = \frac{|c \cup h - c \cap h|}{|\{A,B,C,D\}|}$$

Millionaire 2.0 (two correct choices)

C = {{A, B}, {A, C}, {A, D}, {B, C}, {B, D}, {C, D}}

c = {A, B}

$OptTSets_1(c, \boldsymbol{C})$ = { {(A, True)}, {(B, True)}
{(C, False)}, {(D, False)} }

$OptTSets_2(c, \boldsymbol{C})$ = MinTSets(c, ***C***)
= { {(A, True), (B, True)},
{(C, False), (D, False)} }

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	{A,C}	T	F	T	F	2/4
	{A,D}	T	F	F	T	2/4
	{B,C}	F	T	T	F	2/4
	{B,D}	F	T	F	T	2/4
	{C,D}	F	F	T	T	4/4

$$Err(c,h) := \frac{|c \Delta h|}{|X|} = \frac{|c \cup h - c \cap h|}{|\{A,B,C,D\}|}$$

Err(c, {A,C}) = 2/4

two correct choices)

{A, D}, {B, C}, {B, D}, {C, D}}

c = {A, B}

OptTSets$_1$(c, ***C***) = { {(A, True)}, {(B, True)}
{(C, False)}, {(D, False)} }

OptTSets$_2$(c, ***C***) = MinTSets(c, ***C***)
= { {(A, True), (B, True)},
{(C, False), (D, False)} }

Optimal Teaching Error

Definition

Worst case error

$$OptTErr_k(c,C) := \min_{S:|S|\le k} \boxed{\max_{h\in CONS(S,C)} Err(c,h)}$$

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	h	A	B	C	D	Err(c, h)
c =	{A,B}	T	T	F	F	0/4
	{A,C}	T	F	T	F	2/4
	{A,D}	T	F	F	T	2/4
	{B,C}	F	T	T	F	2/4
	{B,D}	F	T	F	T	2/4
	{C,D}	F	F	T	T	4/4

Err(c, {C,D}) = 4/4

$$Err(c,h) := \frac{|c\Delta h|}{|X|} = \frac{|c\cup h - c\cap h|}{|\{A,B,C,D\}|}$$

Millionaire 2.0 (two correct choices)

C = {{A, B}, {A, C}, {A, D}, {B, C}, {B, D}, {C, D}}

c = {A, B}

OptTSets$_1$(c, ***C***) = { {(A, True)}, {(B, True)}
{(C, False)}, {(D, False)} }

OptTSets$_2$(c, ***C***) = MinTSets(c, ***C***)
= { {(A, True), (B, True)},
{(C, False), (D, False)} }

Optimal Teaching Error

Definition

Worst case error

$$OptTErr_k(c, C) := \min_{S:|S|\le k} \boxed{\max_{h \in CONS(S,C)} Err(c,h)}$$

k-optimal teaching sets achieving the optimal teaching error

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	h	A	B	C	D	Err(c, h)
c =	{A,B}	T	T	F	F	0/4
	{A,C}	T	F	T	F	2/4
	{A,D}	T	F	F	T	2/4
	{B,C}	F	T	T	F	2/4
	{B,D}	F	T	F	T	2/4
	{C,D}	F	F	T	T	4/4

$$Err(c,h) := \frac{|c \Delta h|}{|X|} = \frac{|c \cup h - c \cap h|}{|\{A,B,C,D\}|}$$

Worst case error = 2/4
if teaching (A, True)
(It is optimal)

rrect choices)

}, {B, C}, {B, D}, {C, D}}

OptTSets$_1$(c, ***C***) = { $\boxed{\text{\{(A, True)\}}}$, {(B, True)}
{(C, False)}, {(D, False)} }

OptTSets$_2$(c, ***C***) = MinTSets(c, ***C***)
= { {(A, True), (B, True)},
{(C, False), (D, False)} }

Optimal Teaching Error

Definition

Worst case error

$$OptTErr_k(c,C) := \min_{S:|S|\le k} \boxed{\max_{h \in CONS(S,C)} Err(c,h)}$$

k-optimal teaching sets achieving the optimal teaching error

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	h	A	B	C	D	Err(c, h)
c =	{A,B}	T	T	F	F	0/4
	{A,C}	T	F	T	F	2/4
	{A,D}	T	F	F	T	2/4
	{B,C}	F	T	T	F	2/4
	{B,D}	F	T	F	T	2/4
	{C,D}	F	F	T	T	4/4

$$Err(c,h) := \frac{|c \Delta h|}{|X|} = \frac{|c \cup h - c \cap h|}{|\{A,B,C,D\}|}$$

Millionaire 2.0 (two correct choices)

C = {{A, B}, {A, C}, {A, D}, {B, C}, {B, D}, {C, D}}

c = {A, B}

Worst case error = 4/4
if teaching (A, False)
(It is NOT optimal)

A, True)}, {(B, True)}

C, False)}, {(D, False)} }

$OptTSets_2$(c, ***C***) = MinTSets(c, ***C***)

= { {(A, True), (B, True)},

{(C, False), (D, False)} }

Features of our model (1/2)

Theorem

$$\exists \mathbf{C},\ \exists c \in \mathbf{C},\ \exists k > 0,$$
$$\forall S \in MinTSets(c, \mathbf{C}),\ \ \forall S' \subseteq S,\ \ S' \notin OptTSets_k(c, \mathbf{C}),$$

– Special teaching strategies are necessary
  - A subset of a minimum teaching set is not always optimal

(Proof)

	h	A	B	C	Err(c, h)
c =	{A,B,C}	T	T	T	0/3
	{A,C}	T	F	T	1/3
	{A}	T	F	F	2/3
	{B,C}	F	T	T	1/3
	{B}	F	T	F	2/3

C = {{A, B, C}, {A, C},
{A}, {B, C}, {B}}

c = {A, B, C}

k = 1

MinTSets(c, ***C***) = { {(A, True), (B, True)} }

OptTSets$_1$(c, ***C***) = { {(C, True)} }

Features of our model (1/2)

Theorem

$$\exists \mathbf{C},\ \exists c \in \mathbf{C},\ \exists k > 0,$$
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	h	A	B	C	Err(c, h)
c =	{A,B,C}	T	T	T	0/3
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	{A}	T	F	F	2/3
	{B,C}	F	T	T	1/3
	{B}	F	T	F	2/3

C = {{A, B, C}, {A, C},
{A}, {B, C}, {B}}

c = {A, B, C}

k = 1

Minimum set to teach c

MinTSets(c, ***C***) = { $\boxed{\text{\{(A, True), (B, True)\}}}$ }

OptTSets$_1$(c, ***C***) = { {(C, True)} }

Features of our model (1/2)

Theorem

$$\exists \mathbf{C},\ \exists c \in \mathbf{C},\ \exists k > 0,$$
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(Proof)

	h	A	B	C	Err(c, h)
c =	{A,B,C}	T	T	T	0/3
	{A,C}	T	F	T	1/3
	{A}	T	F	F	$\boxed{2/3}$
	{B,C}	F	T	T	1/3
	{B}	F	T	F	2/3

Worst case error = 2/3

C = {{A, B, C}, {A, C},
{A}, {B, C}, {B}}

k = 1

MinTSets(c, ***C***) = { $\boxed{\text{{(A, True)}}}$, (B, True)} }
OptTSets_1(c, ***C***) = { {(C, True)} }

Features of our model (1/2)

Theorem

$$\exists \mathbf{C},\ \exists c \in \mathbf{C},\ \exists k > 0,$$
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– Special teaching strategies are necessary

- A subset of a minimum teaching set is not always optimal

(Proof)

	h	A	B	C	Err(c, h)
c =	{A,B,C}	T	$\boxed{\text{T}}$	T	0/3
	{A,C}	T	F	T	1/3
	{A}	T	F	F	2/3
	{B,C}	F	T	T	1/3
	{B}	F	T	F	$\boxed{2/3}$

C = {{A, B, C}, {A, C},
{A}, {B, C}, {B}}

c = {A, B, C}

k = 1

Worst case error = 2/3

MinTSets(c, ***C***) = { {(A, True), $\boxed{\text{(B, True)}}$} }

OptTSets$_1$(c, ***C***) = { {(C, True)} }

Features of our model (1/2)

Theorem

$$\exists \mathbf{C},\ \exists c \in \mathbf{C},\ \exists k > 0,$$
$$\forall S \in MinTSets(c,\mathbf{C}),\ \ \forall S' \subseteq S,\ \ S' \notin OptTSets_k(c,\mathbf{C}),$$

– Special teaching strategies are necessary

- A subset of a minimum teaching set is not always optimal

(Proof)

	h	A	B	C	Err(c, h)
c =	{A,B,C}	T	T	T	0/3
	{A,C}	T	F	T	1/3
	{A}	T	F	F	2/3
	{B,C}	F	T	T	1/3
	{B}	F	T	F	2/3

Worst case error = 1/3 (optimal)

C = {{A, B, C}, {A, C}, {A}, {B, C}, {B}}

c = {A, B, C}

MinTSets(c, ***C***) = { {(A, True), (B, True)} }

OptTSets_1(c, ***C***) = { {(C, True)} }

Features of our model (2/2)

Theorem

$$\exists \mathbf{C},\ \exists c \in \mathbf{C},\ \exists k > 0,$$
$$\forall S \in OptTSets_k(c, \mathbf{C}),\ \ c \notin CONS(S, C)$$

– Smart teachers dare to **tell a lie**

- A k-optimal teaching set can be **inconsistent** with c

(Proof)

	h	A	B	C	D	E	Err(c, h)
c =	{A,B,C,D,E}	T	T	T	T	T	0/5
	{B,C,D,E}	F	T	T	T	T	1/5
	{A,B}	T	T	F	F	F	3/5
	{A,C}	T	F	T	F	F	3/5
	{A,D}	T	F	F	T	F	3/5
	{A,E}	T	F	F	F	T	3/5

C = {{A, B, C, D, E}, {B, C, D, E},
{A, B}, {A, C}, {A, D}, {A, E}}

c = {A, B, C, D, E}

k = 1

$OptTSets_1$(c,***C***) = { {(A, False)} }

CONS({(A, False)}, ***C***) = { {B,C,D,E} }

Features of our model (2/2)

Theorem

$$\exists \mathbf{C},\ \exists c \in \mathbf{C},\ \exists k > 0,$$
$$\forall S \in OptTSets_k(c, \mathbf{C}),\ \ c \notin CONS(S, C)$$

– Smart teachers dare to **tell a lie**

- A k-optimal teaching set can be **inconsistent** with c

(Proof)

	h	A	B	C	D	E	Err(c, h)
c =	{A,B,C,D,E}	T	T	T	T	T	0/5
	{B,C,D,E}	F	T	T	T	T	1/5
	{A,B}	T	T	F	F	F	3/5
	{A,C}	T	F	T	F	F	3/5
	{A,D}	T	F	F	T	F	3/5
	{A,E}	T	F	F	F	T	3/5

Worst case error = 1/5 (optimal)
athough (A, False) is a lie

, D, E},
, {A, E}}
= {A, B, C, D, E}
k = 1

$OptTSets_1(c, \boldsymbol{C}) = \{\ \{(A, False)\}\ \}$
$CONS(\ \{(A, False)\}, \boldsymbol{C}) = \{\ \{B,C,D,E\}\ \}$

Features of our model (2/2)

Theorem

$$\exists \mathbf{C},\ \exists c \in \mathbf{C},\ \exists k > 0,$$
$$\forall S \in OptTSets_k(c, \mathbf{C}),\quad c \notin CONS(S, C)$$

– Smart teachers dare to **tell a lie**

- A k-optimal teaching set can be **inconsistent** with c

(Proof)

	h	A	B	C	D	E	Err(c, h)
c =	{A,B,C,D,E}	T	T	T	T	T	0/5
	{B,C,D,E}	F	T	T	T	T	1/5
	{A,B}	T	T	F	F	F	3/5
	{A,C}	T	F	T	F	F	3/5
	{A,D}	T	F	F	T	F	3/5
	{A,E}	T	F	F	F	T	3/5

C = {{A, B, C, D, E}, {B, C, D, E}, {A, B}, {A, C}, {A, D}, {A, E}}

Worst case error = 3/5
If teaching the truth (A, True)

$OptTSets_1(c, \boldsymbol{C})$ = { {(A, False)} }

CONS({(A, False)}, ***C***) = { {B,C,D,E} }

Optimally Incremental Teachability

Definition

c is optimally incrementally teachable w.r.t. ***C***

$$\Updownarrow \; def$$

$$\exists \langle z_1, \ldots, z_{TD(c,\mathbf{C})} \rangle, \; \forall k \in [1, TD(c, \mathbf{C})],$$

$$\{z_1, \ldots, z_k\} \in OptTSets_k(c, \mathbf{C})$$

– Optimal teaching strategies independent of k

Fact

c of Millionaire 2.0 is opt. inc. teachable w.r.t. ***C***

1-opt. 2-opt.

⟨ (A, True), (B, True) ⟩

Optimal order

Millionaire 2.0 (two correct choices)

C = {{A, B}, {A, C}, {A, D}, {B, C}, {B, D}, {C, D}}

c = {A, B}

OptTSets$_2$(c, ***C***) = MinTSets(c, ***C***)

= { {(A, True), (B, True)}, {(C, False), (D, False)} }

OptTSets$_1$(c, ***C***) = { {(A, True)}, {(B, True)}, {(C, False)}, {(D, False)} }

Concept classes M_n^+, M_n', M_n

- M_n: Monomials
  - A concept is a set of $x \in \{0,1\}^n$ satisfying a monomial

    Ex.) Concepts for monomials on 3 variables

    v_1 : {100, 101, 110, 111} ←Monotone monomial

    $\bar{v}_1 v_2 \bar{v}_3$: {010}

    $v_1 \bar{v}_1$: ϕ (Empty concept)

- M_n': Monomials w/o the empty concept
  - $M_n' := M_n - \{\phi\}$
- M_n^+: Monotone monomials
  - Monomials consisting of only positive literals

Opt. inc. teachability of M_n^+

Theorem

M_n^+ is opt. inc. teachable

(Proof sketch)

The condition for c is satisfied by $\langle z_1, z_2, \ldots \rangle$

$$z_i := \begin{cases} (1^{i-1}01^{n-i}, False) & (i \le \ell) \\ (1^{\ell}0^{n-\ell}, True) & (i > \ell) \end{cases}$$

ℓ: # of variables of the target monomial

Ex.) $c \in M_3^+ : v_1 v_2$

z_1 = (0 1 1, False)
z_2 = (1 0 1, False)
z_3 = (1 1 0, True)

Optimal order

Minimum teaching set

$h \in M_3^+$	Err(c, h)
(All true)	$6/2^3$
v_1	$2/2^3$
v_2	$2/2^3$
v_3	$4/2^3$
c → $v_1 v_2$	$0/2^3$
$v_2 v_3$	$2/2^3$
$v_1 v_3$	$2/2^3$
$v_1 v_2 v_3$	$1/2^3$

Opt. inc. teachability of M_n^+

Theorem

M_n^+ is opt. inc. teachable

(Proof sketch)

The condition for c is satisfied by $\langle z_1, z_2, \ldots \rangle$

$$z_i := \begin{cases} (1^{i-1}01^{n-i}, False) & (i \le \ell) \\ (1^{\ell}0^{n-\ell}, True) & (i > \ell) \end{cases}$$

ℓ: # of variables of the target monomial

Exist

Ex.) $c \in M_3^+ : \boxed{v_1} v_2$

$z_1 = (\boxed{0}\ 1\ 1,\ \text{False})$
$z_2 = (1\ 0\ 1,\ \text{False})$
$z_3 = (1\ 1\ 0,\ \text{True})$

Optimal order

Minimum teaching set

	$h \in M_3^+$	Err(c, h)	
	~~(All true)~~	~~$6/2^3$~~	by z_1
	v_1	$2/2^3$	
	~~v_2~~	~~$2/2^3$~~	by z_1
	~~v_3~~	~~$4/2^3$~~	by z_1
c →	v_1v_2	$0/2^3$	
	~~v_2v_3~~	~~$2/2^3$~~	by z_1
	v_1v_3	$2/2^3$	
	$v_1v_2v_3$	$1/2^3$	

Opt. inc. teachability of M_n^+

Theorem

M_n^+ is opt. inc. teachable

(Proof sketch)

The condition for c is satisfied by $\langle z_1, z_2, \ldots \rangle$

$$z_i := \begin{cases} (1^{i-1}01^{n-i}, False) & (i \le \ell) \\ (1^{\ell}0^{n-\ell}, True) & (i > \ell) \end{cases}$$

ℓ: # of variables of the target monomial

Exist Exist

Ex.) $c \in M_3^+ : \boxed{v_1}\boxed{v_2}$

$z_1 = (\boxed{0}\ 1\ 1,\ False)$

$z_2 = (1\ \boxed{0}\ 1,\ False)$

$z_3 = (1\ 1\ 0,\ True)$

Optimal order

Minimum teaching set

	$h \in M_3^+$	Err(c, h)	
	~~(All true)~~	~~$6/2^3$~~	by z_1
	~~v_1~~	~~$2/2^3$~~	by z_2
	~~v_2~~	~~$2/2^3$~~	by z_1
	~~v_3~~	~~$4/2^3$~~	by z_1
c $\rightarrow$	v_1v_2	$0/2^3$	
	~~v_2v_3~~	~~$2/2^3$~~	by z_1
	~~v_1v_3~~	~~$2/2^3$~~	by z_2
	$v_1v_2v_3$	$1/2^3$	

Opt. inc. teachability of M_n^+

Theorem

M_n^+ is opt. inc. teachable

(Proof sketch)

The condition for c is satisfied by ⟨z_1, z_2,…⟩

$$z_i := \begin{cases} (1^{i-1}01^{n-i}, False) & (i \le \ell) \\ (1^{\ell}0^{n-\ell}, True) & (i > \ell) \end{cases}$$

ℓ: # of variables of the target monomial

Exist Exist Not exist

Ex.) $c \in M_3^+ : \boxed{v_1}\boxed{v_2}\boxed{}$

z_1 = ($\boxed{0}$ 1 1, False)
z_2 = (1 $\boxed{0}$ 1, False)
z_3 = (1 1 $\boxed{0}$, True)

Optimal order

Minimum teaching set

$h \in M_3^+$	Err(c, h)	
~~(All true)~~	~~$6/2^3$~~	by z_1
~~v_1~~	~~$2/2^3$~~	by z_2
~~v_2~~	~~$2/2^3$~~	by z_1
~~v_3~~	~~$4/2^3$~~	by z_1
c → v_1v_2	$0/2^3$	
~~v_2v_3~~	~~$2/2^3$~~	by z_1
~~v_1v_3~~	~~$2/2^3$~~	by z_2
~~$v_1v_2v_3$~~	~~$1/2^3$~~	by z_3

Opt. inc. teachability of M_n'

Theorem

M_n' is **not** opt. inc. teachable

(Proof sketch)

z_{opt} := $(0^n$, False) is a special teaching strategy when k=1

Not all negated

Ex.) $c \in M_3'$: $\boxed{v_1 v_2}$

4/2^3

z_{opt} = $\boxed{(0\ 0\ 0,\ \text{False})}$

1-optimal teaching set

Not subset

z_0 = (1 1 1, True)
z_1 = (0 1 1, False)
z_2 = (1 0 1, False)
z_3 = (1 1 0, True)

Minimum teaching set

$h \in M_3'$	Err(c, h)	$h \in M_3'$	Err(c, h)
(All true)	$6/2^3$	$\overline{v_1}$	$6/2^3$
v_1	$2/2^3$	$\overline{v_2}$	$6/2^3$
v_2	$2/2^3$	$\overline{v_3}$	$4/2^3$
v_3	$4/2^3$	$\overline{v_1}v_2$	$4/2^3$
c → v_1v_2	$0/2^3$	$v_1\overline{v_2}$	$4/2^3$
v_2v_3	$2/2^3$	$\overline{v_1}\,\overline{v_2}$	$4/2^3$
v_1v_3	$2/2^3$	$\overline{v_2}v_3$	$4/2^3$
$v_1v_2v_3$	$1/2^3$	…	…

Opt. inc. teachability of M_n'

Theorem

M_n' is **not** opt. inc. teachable

(Proof sketch)

z_{opt} := (0^n, False) is a special teaching strategy when k=1

Not negated

Ex.) $c \in M_3'$: $v_1 v_2$

$4/2^3$

z_{opt} = (0 0 0, False)

1-optimal teaching set

Not subset

z_0 = (1 1 1, True)
z_1 = (0 1 1, False)
z_2 = (1 0 1, False)
z_3 = (1 1 0, True)

For M_n^+

Minimum teaching set

$h \in M_3'$	Err(c, h)	$h \in M_3'$	Err(c, h)
(All true)	$6/2^3$	$\overline{v_1}$	$6/2^3$
v_1	$2/2^3$	$\overline{v_2}$	$6/2^3$
v_2	$2/2^3$	$\overline{v_3}$	$4/2^3$
v_3	$4/2^3$	$\overline{v_1}v_2$	$4/2^3$
c → v_1v_2	$0/2^3$	$v_1\overline{v_2}$	$4/2^3$
v_2v_3	$2/2^3$	$\overline{v_1}\,\overline{v_2}$	$4/2^3$
v_1v_3	$2/2^3$	$\overline{v_2}v_3$	$4/2^3$
$v_1v_2v_3$	$1/2^3$	…	…

Opt. inc. teachability of M_n'

Theorem

M_n' is **not** opt. inc. teachable

(Proof sketch)

z_{opt} := (0^n, False) is a special teaching strategy when k=1

Not negated

Ex.) $c \in M_3'$: $\boxed{v_1 v_2}$

$4/2^3$

z_{opt} = (0 0 0, False)

1-optimal

Can't exclude (All true)

Not subset

z_0 = (1 1 1, True)
z_1 = (0 1 1, False)
z_2 = (1 0 1, False)
z_3 = (1 1 0, True)

For M_n^+

Minimum teaching set

$h \in M_3'$	Err(c, h)	$h \in M_3'$	Err(c, h)
(All true)	$\boxed{6/2^3}$	~~$\overline{v_1}$~~	~~$6/2^3$~~
v_1	$2/2^3$	~~$\overline{v_2}$~~	~~$6/2^3$~~
v_2	$2/2^3$	~~$\overline{v_3}$~~	~~$4/2^3$~~
v_3	$4/2^3$	~~$\overline{v_1}v_2$~~	~~$4/2^3$~~
c → v_1v_2	$0/2^3$	~~$v_1\overline{v_2}$~~	~~$4/2^3$~~
v_2v_3	$2/2^3$	~~$\overline{v_1}\,\overline{v_2}$~~	~~$4/2^3$~~
v_1v_3	$2/2^3$	~~$\overline{v_2}v_3$~~	~~$4/2^3$~~
$v_1v_2v_3$	$1/2^3$	~~…~~	~~…~~

Opt. inc. teachability of M_n'

Theorem

M_n' is **not** opt. inc. teachable

(Proof sketch)

z_{opt} := (0^n, False) is a special teaching strategy when k=1

Not negated

4/2^3

Ex.) $c \in M_3'$: $\boxed{v_1 v_2}$

z_{opt} = $\boxed{(0\ 0\ 0,\ \text{False})}$

Not subset

1-optimal

Can't exclude (All true)

Can't exclude $\overline{v_2}$

z_0 = (1 1 1, e)

z_1 = (0 1 1, False)

z_2 = (1 0 1, False)

z_3 = (1 1 0, True)

For M_n^+

Minimum teaching set

h∈M_3'	Err(c, h)	h∈M_3'	Err(c, h)
~~(All true)~~	~~6/2^3~~	~~$\overline{v_1}$~~	~~6/2^3~~
v_1	2/2^3	$\overline{v_2}$	$\boxed{6/2^3}$
~~v_2~~	~~2/2^3~~	$\overline{v_3}$	4/2^3
~~v_3~~	~~4/2^3~~	~~$\overline{v_1}v_2$~~	~~4/2^3~~
c → v_1v_2	0/2^3	$v_1\overline{v_2}$	4/2^3
~~v_2v_3~~	~~2/2^3~~	$\overline{v_1}\,\overline{v_2}$	4/2^3
v_1v_3	2/2^3	$\overline{v_2}v_3$	4/2^3
$v_1v_2v_3$	1/2^3	…	…

Opt. inc. teachability of M_n'

Theorem

M_n' is **not** opt. inc. teachable

(Proof sketch)

z_{opt} := (0^n, False) is a special teaching strategy when k=1

Not negated

Ex.) $c \in M_3'$: $\boxed{v_1 v_2}$ 4/2³

z_{opt} = $\boxed{(0\ 0\ 0,\ \text{False})}$

1-optimal

Can't exclude (All true)

Can't exclude $\overline{v_2}$

Can't exclude $\overline{v_1}$

Not subset

z_0 = (1 1 1,

z_1 = (0 1 1,

z_2 = (1 0 1, False)

z_3 = (1 1 0, True)

For M_n^+

Minimum teaching set

$h \in M_3'$	Err(c, h)	$h \in M_3'$	Err(c, h)
~~(All true)~~	~~$6/2^3$~~	$\overline{v_1}$	$\boxed{6/2^3}$
~~v_1~~	~~$2/2^3$~~	~~$\overline{v_2}$~~	~~$6/2^3$~~
v_2	$2/2^3$	$\overline{v_3}$	$4/2^3$
~~v_3~~	~~$4/2^3$~~	$\overline{v_1}v_2$	$4/2^3$
c → v_1v_2	$0/2^3$	~~$v_1\overline{v_2}$~~	~~$4/2^3$~~
v_2v_3	$2/2^3$	$\overline{v_1}\,\overline{v_2}$	$4/2^3$
~~v_1v_3~~	~~$2/2^3$~~	~~$\overline{v_2}v_3$~~	~~$4/2^3$~~
$v_1v_2v_3$	$1/2^3$	…	…

Opt. inc. teachability of M_n'

Theorem

M_n' is **not** opt. inc. teachable

(Proof sketch)

z_{opt} := $(0^n$, False$)$ is a special teaching strategy when k=1

Not negated

Ex.) $c \in M_3'$: $\boxed{v_1 v_2}$

4/2^3

z_{opt} = $\boxed{(0\ 0\ 0,\ \text{False})}$

1-optimal

Can't exclude (All true)

Can't exclude $\overline{v_2}$

Can't exclude $\overline{v_1}$

Can't exclude (All true)

Not subset

z_0 = ($\boxed{1\ 1\ 1}$, ...)

z_1 = ($\boxed{0\ 1\ 1}$, ...)

z_2 = ($\boxed{1\ 0\ 1}$, ...)

z_3 = ($\boxed{1\ 1\ 0}$, True)

For M_n

Minimum teaching set

c → $v_1 v_2$

$h \in M_3'$	Err(c, h)	$h \in M_3'$	Err(c, h)
(All true)	$\boxed{6/2^3}$	~~$\overline{v_1}$~~	~~$6/2^3$~~
v_1	$2/2^3$	~~$\overline{v_2}$~~	~~$6/2^3$~~
v_2	$2/2^3$	$\overline{v_3}$	$4/2^3$
~~v_3~~	~~$4/2^3$~~	~~$\overline{v_1} v_2$~~	~~$4/2^3$~~
$v_1 v_2$	$0/2^3$	~~$v_1 \overline{v_2}$~~	~~$4/2^3$~~
~~$v_2 v_3$~~	~~$2/2^3$~~	~~$\overline{v_1}\,\overline{v_2}$~~	~~$4/2^3$~~
~~$v_1 v_3$~~	~~$2/2^3$~~	~~$\overline{v_2} v_3$~~	~~$4/2^3$~~
~~$v_1 v_2 v_3$~~	~~$1/2^3$~~	…	…

Opt. inc. teachability of M_n'

Theorem

M_n' is **not** opt. inc. teachable

Theorem

M_n is **not** opt. inc. teachable

(Proof sketch)

z_{opt} := $(0^n$, False) is a special teaching strategy when k=1

Not negated

4/2^3

Ex.) $c \in M_3'$: $\boxed{v_1 v_2}$

Not subset

z_{opt} = (0 0 0, False)

1-optimal

Can't exclude (All true)

Can't exclude $\overline{v_2}$

Can't exclude $\overline{v_1}$

Can't exclude (All true)

z_0 = (1 1 1

z_1 = (0 1 1

z_2 = (1 0 1

z_3 = (1 1 0, True)

For M_n

Minimum teaching set

$h \in M_3'$	Err(c, h)	$h \in M_3'$	Err(c, h)
(All true)	$\boxed{6/2^3}$	~~$\overline{v_1}$~~	~~$6/2^3$~~
v_1	$2/2^3$	~~$\overline{v_2}$~~	~~$6/2^3$~~
v_2	$2/2^3$	$\overline{v_3}$	$4/2^3$
~~v_3~~	~~$4/2^3$~~	~~$\overline{v_1}v_2$~~	~~$4/2^3$~~
c → v_1v_2	$0/2^3$	~~$v_1\overline{v_2}$~~	~~$4/2^3$~~
~~v_2v_3~~	~~$2/2^3$~~	~~$\overline{v_1}\,\overline{v_2}$~~	~~$4/2^3$~~
~~v_1v_3~~	~~$2/2^3$~~	~~$\overline{v_2}v_3$~~	~~$4/2^3$~~
~~$v_1v_2v_3$~~	~~$1/2^3$~~	…	…

Interesting result

Theorem

$$\forall k \in [4, 2^{n-1} - 1],$$
$$\forall S \in OptTSets_k(\phi, M_n), \quad \phi \notin \mathrm{CONS}(\mathrm{S}, \mathrm{M_n})$$

– Teachers must tell a lie to **optimally** teach ϕ in M_n

Theorem ([Goldman and Kearns 1995])

$$TD(c, M_n) = \min\{\ell + 2, \mathrm{n} + 1\}$$

(Proof sketch when k>n)

$c' \in M_n : v_1 v_2 \ldots v_n$

TD(c', M_n) = n+1

	h∈M_n	Err(c, h)
c =	ϕ	$0/2^n$
	…	…
c' =	$v_1 v_2 \ldots v_n$	$1/2^n$
	…	…

S∈MinTSets(c', M_n) is k-optimal for c
However, S is inconsistent with c

Theorem

$$\exists \mathbf{C},\ \exists c \in \mathbf{C},\ \exists k > 0,$$
$$\forall S \in OptTSets_k(c, \mathbf{C}),\quad c \notin CONS(S, C)$$

Natural example

Theorem

$$\forall k \in [4, 2^{n-1} - 1],$$
$$\forall S \in OptTSets_k(\phi, M_n),\quad \phi \notin \mathrm{CONS}(\mathrm{S}, \mathrm{M_n})$$

– Teachers must tell a lie to **optimally** teach ϕ in M_n

Theorem ([Goldman and Kearns 1995])

$$TD(c, M_n) = \min\{\ell + 2, \mathrm{n} + 1\}$$

(Proof sketch when k>n)

$$c' \in M_n :\ v_1 v_2 \ldots v_n$$

TD(c', M_n) = n+1

$h \in M_n$	Err(c, h)
c = ϕ	$0/2^n$
…	…
c' = $v_1 v_2 \ldots v_n$	$1/2^n$
…	…

S∈MinTSets(c', M_n) is k-optimal for c

However, S is inconsistent with c

Theorem

$$\exists \mathbf{C},\ \exists c \in \mathbf{C},\ \exists k > 0,$$
$$\forall S \in OptTSets_k(c, \mathbf{C}),\quad c \notin CONS(S, C)$$

Natural example

Theorem

$$\forall k \in [4, 2^{n-1} - 1],$$
$$\forall S \in OptTSets_k(\phi, M_n),\quad \phi \notin \mathrm{CONS}(\mathrm{S}, \mathrm{M_n})$$

– Teachers must tell a lie to **optimally** teach ϕ in M_n

(Proof sketch when k>n)

Theorem ([Goldman and Kearns 1995])

$$TD(c, M_n) = \min\{\ell + 2, \mathrm{n} + 1\}$$

$$c' \in M_n :\ v_1 v_2 \ldots v_n$$

TD(c', M_n) = n+1

	$h \in M_n$	Err(c, h)
c =	~~ϕ~~	~~$0/2^n$~~
	~~…~~	~~…~~
c' =	$v_1 v_2 \ldots v_n$	$1/2^n$
	~~…~~	~~…~~

S ∈ MinTSets(c', M_n) is k-optimal for c
However, S is inconsistent with c

Summary

	M_n^+	M_n'	M_n
Teaching Dim., TD(C) [Goldman and Kearns 1991]	n	n+1	2^n
Teachability [Shinohara and Miyano 1991]	True	True	False
Opt. Teaching Error $\mathrm{OptTE}_k(C)$	$\frac{2^{n-k}-1}{2^n}$	$\frac{2^{n-k+1}-1}{2^n}$	$\begin{cases} \frac{2^{n-k+1}-1}{2^n} & (k \le 2) \\ \frac{2^{n-k+1}}{2^n} & (2 < k \le n) \\ \frac{1}{2^n} & (n < k < 2^n) \end{cases}$
Opt. Inc. Teachability	True	False	False

Our results

Quite small

Different boundary

Thank you for your attention!